# Technical Notes

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## Simplification of Beam and Warming's Implicit Scheme for Two-Dimensional Compressible Flows

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### Introduction

THE viscous flux vectors have been found decomposable to the Jacobian matrix and flow variable vector. We shall show how the decomposition of flux vectors can be used to either simplify or alter the implicit factored scheme for the compressible two-dimensional Navier-Stokes equations, originally derived by Beam and Warming in 1978.[1] The modifications proposed here are localized in the implicit time-difference terms involving the Jacobian matrices.

### Decomposition of Flux Vectors

Using the notations of Beam and Warming[1] (called, hereafter, B-W), we begin with the two-dimensional Navier-Stokes equations in the conservative form

$$\frac{\partial U}{\partial t}+\frac{\partial F}{\partial x}+\frac{\partial G}{\partial y}=\frac{\partial V_1}{\partial x}+\frac{\partial V_2}{\partial x}+\frac{\partial W_1}{\partial y}+\frac{\partial W_2}{\partial y} \tag{1}$$

Here, $U$ is the column vector of flow variables,

$$U=\begin{bmatrix}\rho\\ \rho u\\ \rho v\\ e\end{bmatrix}$$

the flux vectors $F$ and $G$ refer to the inertial terms,

$$F=\begin{bmatrix}\rho u\\ \rho u^2+p\\ \rho uv\\ u(e+p)\end{bmatrix},\qquad G=\begin{bmatrix}\rho v\\ \rho uv\\ \rho v^2+p\\ v(e+p)\end{bmatrix}$$

and the shear stress terms are grouped into the following viscous flux vectors,

$$V_1=\begin{bmatrix}0\\ (2\mu+\lambda)u_x\\ \mu v_x\\ (2\mu+\lambda)uu_x+\mu vu_x+kT_x\end{bmatrix},\qquad V_2=\begin{bmatrix}0\\ \lambda v_y\\ \mu u_y\\ \lambda uv_y+\mu vu_y\end{bmatrix}$$

$$W_1=\begin{bmatrix}0\\ \mu v_x\\ \lambda u_x\\ \mu uv_x+\lambda vu_x\end{bmatrix},\qquad W_2=\begin{bmatrix}0\\ \mu u_y\\ (2\mu+\lambda)v_y\\ \mu uu_y+(2\mu+\lambda)vv_y+kT_y\end{bmatrix}$$

where $p=(\gamma-1)[e-(\rho u^2+\rho v^2)/2]$ and $T=[(e/\rho)-(u^2+v^2)/2]/C_v$. The flux vectors as given in the Appendix of B-W[1] have been reproduced here for the convenience of the reader.

By a tedious but straightforward manipulation, one can compute the $4\times 4$ Jacobians $A=(\partial F/\partial U)$ and $B=(\partial G/\partial U)$ to show that

$$F=AU,\qquad G=BU \tag{2}$$

which Beam and Warming[2] first used in the implicit time-difference scheme for the two-dimensional Euler equations. Similarly, the viscous flux vectors have the following decompositions

$$V_1=-PU,\quad V_2=-TU,\quad W_1=-YU,\quad W_2=-QU \tag{3}$$

where $P=(\partial V_1/\partial U)$, $T=(\partial V_2/\partial U)$, $Y=(\partial W_1/\partial U)$, and $Q=(\partial W_2/\partial U)$. These decompositions are completely unexpected [at least to me, although R. M. Beam has informed me that he was aware of Eqs. (3) at the time of the writing of Ref. 1]. In fact, they are of the curious identity that Beam and Warming have commented on in a footnote in Ref. 2 (p. 91). Note that $F$ and $G$ are homogeneous functions of degree 1 in $U$, whereas $V$ and $W$ are homogeneous functions of degree $-1$ in $U$ (see, for example, Ref. 3).

For $V_1$ and $W_2$, a corollary of Eqs. (3) is found in $U_x$ and $U_y$, i.e.,

$$V_1=RU_x,\qquad W_2=SU_y \tag{4}$$

where $R=(\partial V_1/\partial U_x)$ and $S=(\partial W_2/\partial U_y)$. It must be pointed out that Eqs. (3) and (4) are the conjugate forms, as may be checked by the following identities. First, by denoting the element-wise derivatives of $R$ and $S$ with respect to $x$ and $y$ by $R_x$ and $S_y$, we find that

$$R_x=R_{(tc,x)}+P,\qquad S_y=S_{(tc,y)}+Q \tag{5}$$

Here, the subscript $(tc,x)$ signifies that $\partial/\partial x$ operates on the transport coefficients ($\mu$, $\lambda$, $k$) only and not on flow variables; the subscript $(tc,y)$ is interpreted similarly. Note that Eqs. (5) are a restatement of B-W's equations (A13) and (A14). Second, we further find that

$$RU=0,\qquad SU=0 \tag{6a}$$

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together with

$$R_{(tc,x)}U = 0, \qquad S_{(tc,y)}U = 0 \tag{6b}$$

With the use of identities (5) and (6), one can show that $V_1 = -PU$ and $V_1 = RU_x$ are one and the same representation, and so are $W_2 = -QU$ and $W_2 = SU_y$.

## Time-Difference Schemes

For the time discretization $U^n = U(n\Delta t)$, it is necessary to obtain an equation for $\Delta U^n = U^{n+1} - U^n$ by taking the difference of Eq. (1) written for $t = (n+1)\Delta t$ and $n\Delta t$,

$$\frac{\partial}{\partial t}\Delta U^n + \frac{\partial}{\partial x}\Delta F^n + \frac{\partial}{\partial y}\Delta G^n - \frac{\partial}{\partial x}\Delta V_1^n - \frac{\partial}{\partial y}\Delta W_2^n = \frac{\partial}{\partial x}\Delta V_2^n + \frac{\partial}{\partial y}\Delta W_1^n \tag{7}$$

where $\Delta F^n = F^{n+1} - F^n$, etc. By Taylor expansion, B-W obtained the second-order time-accurate approximation for the flux vectors in the left side of Eq. (7).

$$\Delta F^n = A^n \Delta U^n + \mathcal{O}(\Delta t^2) \tag{8a}$$

$$\Delta G^n = B^n \Delta U^n + \mathcal{O}(\Delta t^2) \tag{8b}$$

$$\Delta V_1^n = (P - R_x)^n \Delta U^n + \frac{\partial}{\partial x}(R\Delta U)^n + \mathcal{O}(\Delta t^2) \tag{8c}$$

$$\Delta W_2^n = (Q - S_y)^n \Delta U^n + \frac{\partial}{\partial y}(S\Delta U)^n + (\Delta t^2) \tag{8d}$$

as given by Eq. (4) of B-W.[1] Inserting Eq. (7) together with approximations (8) into the single-step time-difference scheme, B-W have obtained the implicit finite difference formulation (yet unfactored),

$$\left\{I + \frac{\theta\Delta t}{1+\xi}\left[\frac{\partial}{\partial x}(A - P + R_x)^n - \frac{\partial^2}{\partial x^2}(R)^n + \frac{\partial}{\partial y}(B - Q + S_y)^n - \frac{\partial^2}{\partial y^2}(S)^n\right]\right\}\Delta U^n = \text{RHS of B-W's Eq. (6)} \tag{9}$$

It is, however, possible to simplify Eqs. (8c) and (8d) with the use of Eqs. (5) and (6); that is,

$$\Delta V_1^n = -R_{(tc,x)}^n U^{n+1} + \frac{\partial}{\partial x}(RU^{n+1}) \tag{10a}$$

$$\Delta W_2^n = -S_{(tc,y)}^n U^{n+1} + \frac{\partial}{\partial y}(S^n U^{n+1}) \tag{10b}$$

Since $\Delta V_1^n$ and $\Delta W_2^n$ have no explicit contribution, the left side of Eq. (9) reduces to

$$\left\{I + \frac{\theta\Delta t}{1+\xi}\left[\frac{\partial}{\partial x}(A + R_{(tc,x)})^n - \frac{\partial^2}{\partial x^2}(R)^n + \frac{\partial}{\partial y}(B + S_{(tc,y)})^n - \frac{\partial^2}{\partial y^2}(S)^n\right]\right\}U^{n+1} - \left\{I + \frac{\theta\Delta t}{1+\xi}\left[\frac{\partial}{\partial x}(A)^n + \frac{\partial}{\partial y}(B)^n\right]\right\}U^n = \text{RHS of B-W's Eq. (6)} \tag{11}$$

Note that $P$ and $Q$ do not appear in Eq. (11), and $U^{n+1}$ and $U^n$ no longer have the same coefficients.

Let us now derive alternate expressions for $\Delta V_1$ and $\Delta W_2$ via decompositions (3). First, we have from $V_1 = -PU$

$$(\Delta V_1^n)_{\text{ex}} = -P^{n+1}U^{n+1} + P^n U^n \tag{12}$$

where the subscript ex signifies that it is exact. Inserting $P^nU^{n+1} - P^nU^{n+1}$ into Eq. (12) yields

$$(\Delta V_1^n)_{\text{ex}} = -P^n\Delta U^n - \Delta P^n U^{n+1} \tag{13}$$

Although the first term on the right side of Eq. (13) is suitable for the construction of the implicit finite difference scheme, the second term is not because $\Delta P^n$ introduces implicit coefficients for $U^{n+1}$. We shall, therefore, replace the second term $\Delta P^n U^{n+1}$ by $\Delta P^{n-1}U^n$; i.e.,

$$\Delta V_1^n = -P^n \Delta U^n - \Delta P^{n-1}U^n + \epsilon \tag{14}$$

where $\epsilon$ is the error due to replacement. Since $\epsilon = \mathcal{O}(\Delta t^2)$, as shown in the Appendix, the use of Eq. (14) is numerically comparable to that of Eq. (8c). By writing out Eq. (14), we therefore obtain the working expression

$$\Delta V_1^n = -P^n U^{n+1} + P^{n-1}U^n + \mathcal{O}(\Delta t^2) \tag{15a}$$

which differs from Eq. (12) by shifting down the time index for $P$ by 1. Similarly,

$$\Delta W_2^n = -Q^n U^{n+1} + Q^{n-1}U^n + \mathcal{O}(\Delta t^2) \tag{15b}$$

With the use of Eqs. (15), we arrive at the alternate formulation

$$\left\{I + \frac{\theta\Delta t}{1+\xi}\left[\frac{\partial}{\partial x}(A + P)^n + \frac{\partial}{\partial y}(B + Q)^n\right]\right\}U^{n+1} - \left\{I + \frac{\theta\Delta t}{1+\xi}\left[\frac{\partial}{\partial x}(A^n + P^{n-1}) + \frac{\partial}{\partial y}(B^n + Q^{n-1})\right]\right\}U^n = \text{RHS of B-W's Eq. (6)} \tag{16}$$

## Conclusion

In contrast to Eq. (11), the alternate formulation uses the Jacobians $P$ and $Q$ directly, instead of the derivatives of $R$ and $S$. An obvious drawback is that Eq. (16) is now a three-point formula involving the time indices $n-1$, $n$, and $n+1$ i.e., not a delta form. We have, therefore, traded the spatial differentiation of $R$ and $S$ with the evaluation of backward-time Jacobians $P^{n-1}$ and $Q^{n-1}$. Although the discussion has thus far been restricted to the handling of $V_1$ and $W_2$, it appears that decompositions (3) might allow us to retain $V_2$ and $W_1$ also as implicit contributions. Whether the use of newly found decompositions is simply a mathematical alternative or offers real computational savings is yet to be determined.

## Appendix: Estimating the Replacement Error in Eq. (14)

For $V_1 = -PU$, we evaluate $\Delta V_1^n$ by Taylor expansion

$$\begin{aligned}\Delta V_1^n &= \frac{\partial}{\partial U}(-PU)^n\Delta U^n + \frac{\partial}{\partial U_x}(-PU)^n\Delta U_x^n + \mathcal{O}(\Delta t^2)\\ &= -P^n\Delta U^n - \frac{\partial}{\partial U}(P)^nU^n\Delta U^n - \frac{\partial}{\partial U_x}(P)^nU^n\Delta U_x^n + \mathcal{O}(\Delta t^2)\\ &= -P^n\Delta U^n - \left[\frac{\partial}{\partial U}(P)^n\Delta U^n + \frac{\partial}{\partial U_x}(P)^n\Delta U_x^n\right]U^n + \mathcal{O}(\Delta t^2)\\ &= -P^n\Delta U^n - \Delta P^n U^n + \mathcal{O}(\Delta t^2)\end{aligned} \tag{A1}$$

Note that the quantity in the square brackets has been identified as $\Delta P^n$ in the last equality. By comparing Eqs. (A1) with the exact expression (13), we find that

$$|\Delta P^n \Delta U^n| = \mathcal{O}(\Delta t^2) \quad \text{(A2)}$$

Now, we rearrange the right side of Eq. (14) as

$$\Delta V_1^n - (\Delta V_1^n)_{ex} = \Delta P^n \Delta U^n + (\Delta P^n - \Delta P^{n-1}) U^n \quad \text{(A3)}$$

In view of Eq. (A2) together with $|\Delta P^n - \Delta P^{n-1}| = \mathcal{O}(\Delta t^2)$, we infer that $\epsilon = \mathcal{O}(\Delta t^2)$.

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### References

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# Formation of Shocks Within Axisymmetric Nozzles

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## I. Introduction

THE formation of shocks within axisymmetric supersonic nozzles has received considerable attention in the past, both experimentally and computationally. The presence of undesirable oblique shocks can significantly alter the downstream flowfield, reduce the thrust efficiency, and affect both the external acoustic signature and base pressure. The experimental study by Back and Cuffel[1] described formation of an oblique shock just downstream of the throat for a tested nozzle geometry and documented centerline Mach number distribution based on pitot stagnation pressures. Measured radial pitot stagnation pressure distributions at the nozzle exit and two locations downstream were obtained for a second nozzle exiting into vacuum ambient conditions with a significantly different nozzle contour.[2] Typical past predictions of internal and external nozzle flow include approximate methods,[3] the method of characteristics,[4] and finite difference schemes.[5]

Recently, unstructured adaptive grids in conjunction with conservative nonlinear shock-capturing schemes have been used to yield fine-grid resolution near high flow gradients. The ability of these codes to capture and predict sharp shocks and contact discontinuities has been demonstrated and validated.[6,7] The newly developed axisymmetric finite element method-flux corrected transport (FEM-FCT) scheme[8,9] is used here to investigate the flow and types of shock formation for various nozzle configurations.

## II. Numerical Method

The Euler equations for axisymmetric compressible flow of an ideal gas can be discretized to form a conservative scheme.[7] The two-step second-order Taylor-Galerkin algorithm has been used for the computation of inviscid and viscous flows for the Cartesian[7,8] and axisymmetric[6] coordinate systems. In the first predictor step, the conserved quantitites are assumed piecewise constant, and for the second corrector step, they are assumed piecewise linear. Spatial discretization is performed via the Galerkin weighted residual method, with interpolation conducted separately on the conserved quantities and the radial distance. This leads to a higher accuracy in the $r$ direction[9] and allows a closed-form derivation of the weighted residual statements.

Two types of artificial viscosity are added to the high-order scheme just mentioned (which is essentially fourth-order phase accurate). The first is mass diffusion, which is added to yield a monotonic low-order scheme. The low-order term contribution is combined with the high-order term contribution near admissible discontinuities through the FEM-FCT[7] formulation to prevent the formation of overshoots or undershoots in the conserved quantities. To maintain strict conservation, this limiting is carried out on the element level.[7] The FCT scheme has been shown to be highly robust and accurate for several fluid dynamics problems using both finite difference (see Ref. 10) and finite element schemes.[6-8,11] In addition, the modified Lapidus artificial viscosity, which proved successful for Cartesian coordinate systems, is extended to the axisymmetric case.[6] A Lapidus coefficient of 2 was used, which maintained flux conservation but did extend contact discontinuities for over typically four cells.

Adaptive remeshing was employed to optimize the distribution of grid points by refining areas of high density gradients and coarsening areas of low density gradients. The remeshing allowed a balanced (and efficient) distribution of truncation errors by controlling the relative size of local computational cells. Such remeshing may reduce storage and CPU requirements by 10–100 times in advection-dominated flows as compared to an overall fine grid.[8] The remeshing was accomplished automatically with the advancing front grid generator and typically requires three to four grid adaptations for a steady-state flow solution.

The subsonic inflow boundary conditions employed three characteristic conditions to update boundary values. Along the nozzle wall, tangent flow was imposed such that all fluxes normal to the wall were eliminated. Finally, outflow boundary conditions were set to be free (i.e., no correction on predicted values) for supersonic outflow, which assumes a sufficiently low enough backpressure (which is consistent with the experimental operating conditions). Further boundary condition discussion may be found in Löhner et al.[6] All computations were performed on a Cray X-MP 2/4 using local time stepping and typically took less than half an hour of Cray CPU time to converge.

## III. Results and Discussion

Various internal and external axisymmetric nozzle flowfields were predicted using the FEM-FCT scheme. Figure 1a

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